The essential spectrum of Schrödinger operators on lattices

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Abstract

The paper is devoted to the study of the essential spectrum of discrete Schrödinger operators on the lattice $\mathbb{Z}^N$ by means of the limit operators method. This method has been applied by one of the authors to describe the essential spectrum of (continuous) electromagnetic Schrödinger operators, square-root Klein-Gordon operators, and Dirac operators under quite weak assumptions on the behavior of the magnetic and electric potential at infinity. The present paper is aimed to illustrate the applicability and efficiency of the limit operators method to discrete problems as well.

We consider the following classes of the discrete Schrödinger operators: 1) operators with slowly oscillating at infinity potentials, 2) operators with periodic and semi-periodic potentials; 3) Schrödinger operators which are discrete quantum analogs of the acoustic propagators for waveguides; 4) operators with potentials having an infinite set of discontinuities; and 5) three-particle Schrödinger operators which describe the motion of two particles around a heavy nuclei on the lattice $\mathbb{Z}^3$.

1 Introduction

The present paper deals with applications of the limit operators method to the description of the essential spectrum of several classes of discrete Schrödinger operators. In [18], this method has been applied to study the location of the essential spectrum of electromagnetic Schrödinger operators, square-root Klein-Gordon operators, and Dirac operators under very weak assumptions on the behavior of magnetic and electric potentials at infinity. One remarkable outcome of this approach is a simple and transparent proof of the well known Hunziker, van Winter, Zjislin (HWZ) theorem on the location of the essential spectrum of multi-particle Hamiltonians ([4]).

Roughly speaking, the limit operators approach of [18] works as follows. The study of the essential spectrum of an unbounded operator is reduced to the study of the essential spectrum of a related bounded operator which belongs to a certain Banach algebra $\mathcal{B}$. To each operator $A \in \mathcal{B}$, there is associated

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a family $\{A_h\}$ of operators, called the limit operators of A, which reflect the behaviour of the operator A at infinity. Based on results of [20, 21], it is shown in [18] that

$$sp_{ess}\,(A) = \bigcup sp\,(A_h) \tag{1}$$

where the union is taken over all limit operators A_h of A. In general, the limit operators of a given operator have a simpler structure than the operator itself. Hence, in many instances, (1) provides an effective tool for the description of the essential spectrum of operators in $\mathcal{B}$. An identity similar to (1) holds for operators in the Wiener algebra on $\mathbb{Z}^N$ (see below).

A formula similar to (1) has been obtained independently (but later) in the recent paper [12] by using a localization technique and an appropriate partition of unit on a Hilbert space, and in [3, 7, 6, 14] (see also the references cited therein) by means of C^*-algebra techniques. Note that the methods used in [3, 7, 6, 12, 14] are restricted to the description of the essential spectrum of *selfadjoint or normal* operators acting on a Hilbert space whereas the limit operators approach which will be discussed below works for non-selfadjoint operators acting on L^p-type spaces, for example for Schrödinger operators with complex potentials on L^p-spaces, as well.

In this paper, we consider discrete electromagnetic Schrödinger operators of the form

$$Hu = \sum_{k=1}^{N} \frac{1}{2m_k}(V_{e_k} - a_k I)(V_{-e_k} - \bar{a}_k I)u + \Phi u \tag{2}$$

on the lattice $\mathbb{Z}^N$. Here, $(V_g u)(x) := u(x-g)$, $x \in \mathbb{Z}^N$, is the operator of shift by $g \in \mathbb{Z}^N$, the e_j are the vectors $(0, \ldots, 1, \ldots, 0)$ with the 1 standing at the jth position, m_k is the mass of the kth particle, and the a_k, $k = 1, \ldots, N$ and Φ are bounded complex-valued functions on $\mathbb{Z}^N$. The vector-valued function $a = (a_1, \ldots, a_N)$ can be considered as an analogue of the magnetic potential, whereas Φ is the discrete analogue of the electric potential. If Φ is real-valued, then H is a selfadjoint operator on the Hilbert space $l^2(\mathbb{Z}^N)$.

Operators of the form (2) describe the so-called tight binding model in solid state physics (see [15, 16] and the references given there), which plays a prominent role in the theory of propagation of spin waves and of waves in quasicrystals [25, 23], in the theory of nonlinear integrable lattices [25, 5], and in other places. There is an extensive bibliography devoted to different aspects of the spectral theory of discrete Schrödinger operators. Let us only mention [1, 11, 2, 24, 12, 25, 28], and see also the references cited in these papers.

The present paper is organized as follows. In Section 2, we recall some auxiliary material concerning the Wiener algebra of discrete operators on $\mathbb{Z}^N$ and the limit operators method. Our presentation will follow the papers [20, 21]. A comprehensive account of this material can be found in [22]. In Section 3, we give applications of the limit operators method to the description of the essential spectrum of the discrete Schrödinger operator (2) under different assumptions on the behavior of magnetic and electric potentials at infinity. In particular, we are going to describe the essential spectra for the following classes of potentials.

1) *Slowly oscillating at infinity potentials.* This class of potentials is remarkable, since the limit operators of Schrödinger operators with slowly oscillating potentials are unitarily equivalent to operators of the form

$$\sum_{k=1}^{N} \frac{1}{2m_k} \Delta_k + \Phi_h I, \tag{3}$$

where Φ_h is constant and Δ_k is the discrete Laplacian $2I - V_{e_k} - V_{-e_k}$ with respect to the variable $x_k \in \mathbb{Z}$.

2) *Periodic and semi-periodic potentials.* For periodic operators, the essential spectrum coincides with the spectrum. There exists a matrix realization of (2) which allows one to describe the spectrum explicitly. We prove that the limit operators for Schrödinger operators with semi-periodic potentials are periodic operators. Another approach to one-dimensional periodic Jacobi operators is given in [25].

3) *Quantum waveguides.* We consider the discrete Schrödinger operator $H = \sum_{k=1}^{N} \Delta_k + \Phi I$ which is a discrete quantum analog of the acoustic propagator for waveguides (see [26, 27]). The limiting absorption principle as well as the scattering theory for the discrete acoustic propagator are considered in [17].

4) *Potentials with an infinite set of discontinuities.* We study the one-dimensional Schrödinger operator with electric potential Φ which takes two real values a and b and which has a countable set of discontinuities. More precisely, Φ is equal to a on $\Lambda = \cup_{k=0}^{\infty} \{x \in \mathbb{Z} : \gamma_k^- \leq |x| \leq \gamma_k^+\}$ and equal to b on $\mathbb{R} \setminus \Lambda$ where

$$\lim_{k\to\infty} \gamma_k^- = \lim_{k\to\infty} \gamma_k^+ = \lim_{k\to\infty} (\gamma_k^+ - \gamma_k^-) = \lim_{k\to\infty} (\gamma_k^- - \gamma_{k-1}^+) = +\infty. \tag{4}$$

We prove that $sp_{ess}\,(\Delta + \Phi I) = [a,\, a+4] \cup [b,\, b+4]$ in this case.

5) We consider the three-particle Schrödinger operator H on $l^2(\mathbb{Z}^6)$ which describes the motion of two particles x^1, $x^2 \in \mathbb{Z}^3$ with masses m_1, m_2 on the lattice $\mathbb{Z}^3$ around a heavy nuclei located at the point 0. We prove that

$$sp_{ess}\,(H) = sp\,(H_1) \cup sp\,(H_2) \cup sp\,(H_{12}), \tag{5}$$

where H_j, $j = 1, 2$ is the Hamiltonian of the subsystem consisting of the particle x^j and the nuclei, and where H_{12} is the Hamiltonian of the subsystem consisting of x^1 and x^2. We further apply (5) to estimate the lower and upper bound of the essential spectrum of the three-particle Hamiltonian. Compare also the papers [1, 2, 11] devoted to spectral properties of three-particle problems in the impulse representation.

2 Auxiliary material

2.1 Fredholm theory and essential spectrum of operators in the Wiener algebra

We will use the following notations. For each Banach space X, let $L(X)$ denote the Banach algebra of all bounded linear operators acting on X and $K(X)$ the

ideal of $L(X)$ of all compact operators. An operator $A \in L(X)$ is called a *Fredholm operator* if $\ker A := \{x \in X : Ax = 0\}$ and $\operatorname{coker} A := X/A(X)$ are finite dimensional spaces. The essential spectrum of A consists of all complex numbers λ for which the operator $A - \lambda I : X \to X$ is not Fredholm. We denote the spectrum and the essential spectrum of $A \in L(X)$ by $sp\,(A|X)$ and $sp_{ess}\,(A|X)$, respectively. If the dependence of these spectra on X is evident, we simply write $sp\,(A)$ and $sp_{ess}\,(A)$ for the spectrum and the essential spectum of A. The discrete spectrum of A, denoted by $sp_{dis}\,(A|X)$ or $sp_{dis}\,(A)$, is the set of all isolated eigenvalues of finite multiplicity of A. Note that

$$sp_{dis}\,(A|X) \subseteq sp\,(A|X) \setminus sp_{ess}\,(A|X)$$

and that equality holds in this inclusion if A is self-adjoint. Moreover, if $D \subseteq \mathbb{C} \setminus sp_{ess}\,(A)$ is a domain which contains at least one point which is not in $sp(A)$, then

$$sp\,(A) \cap D = sp_{dis}\,(A) \cap D$$

(Corollary 8.4 in Chapter XI of [8]).

For $p \in [1, \infty)$ and N a positive integer, let $l^p(\mathbb{Z}^N)$ denote the Banach space of all functions $u : \mathbb{Z}^N \to \mathbb{C}$ for which the norm

$$\|u\|_{l^p(\mathbb{Z}^N)} := \left(\sum_{x \in \mathbb{Z}^N} |u(x)|^p \right)^{1/p}$$

is finite, and let $l^\infty(\mathbb{Z}^N)$ refer to the C^*-algebra of all bounded complex-valued functions on $\mathbb{Z}^N$ with norm

$$\|a\|_{l^\infty(\mathbb{Z}^N)} := \sup_{x \in \mathbb{Z}^N} |a(x)|.$$

The following results hold for operators on l^p-spaces of vector-valued sequences and (with modifications) for operators on l^p-spaces of sequences with values in an arbitrary Banach space as well. We will not need these results here and refer the monograph [22] for details.

Let $\{a_\alpha\}_{\alpha \in \mathbb{Z}^N}$ be a sequence of functions in $l^\infty(\mathbb{Z}^N)$. Consider the difference operator

$$(Au)(x) := \sum_{\alpha \in \mathbb{Z}^N} a_\alpha(x)(V_\alpha u)(x), \quad x \in \mathbb{Z}^N, \tag{6}$$

which is well defined for compactly supported functions u. We let $W(\mathbb{Z}^N)$ stand for the class of all operators of the form (6) which satisfy

$$\|A\|_{W(\mathbb{Z}^N)} := \sum_{\alpha \in \mathbb{Z}^N} \|a_\alpha\|_{l^\infty(\mathbb{Z}^N)} < \infty. \tag{7}$$

The set $W(\mathbb{Z}^N)$ provided with standard operations and with the norm (7) forms a Banach algebra, the so-called *Wiener algebra* on $\mathbb{Z}^N$. It is easy to see that this algebra is continuously embedded into $L(l^p(\mathbb{Z}^N))$ for each $p \in [1, \infty]$.

Proposition 1 *Let $p \in [1, \infty]$. The Wiener algebra $W(\mathbb{Z}^N)$ is inverse closed in each of the algebras $L(l^p(\mathbb{Z}^N))$, that is, if $A \in W(\mathbb{Z}^N)$ possesses a bounded inverse A^{-1} on $l^p(\mathbb{Z}^N)$, then $A^{-1} \in W(\mathbb{Z}^N)$.*

Proposition 2 *Let $A \in W(\mathbb{Z}^N)$. Then the spectrum $sp\,(A|l^p(\mathbb{Z}^N))$ does not depend on $p \in [1, \infty]$.*

The following definition introduces our main tool to describe the essential spectra of operators in the Wiener algebra.

Definition 3 *Let $p \in (1, \infty)$, and let $h : \mathbb{N} \to \mathbb{Z}^N$ be a sequence tending to infinity. An operator $A_h \in L(l^p(\mathbb{Z}^N))$ is called the* limit operator of $A \in L(l^p(\mathbb{Z}^N))$ defined by the sequence h *if the strong limits on $l^p(\mathbb{Z}^N)$*

$$A_h = \text{s-lim}_{j\to\infty} V_{-h(j)} A V_{h(j)}, \quad A_h^* = \text{s-lim}_{j\to\infty} V_{-h(j)} A^* V_{h(j)}$$

exist. We write $op\,(A|l^p(\mathbb{Z}^N))$ or simply $op\,(A)$ for the set of all limit operators of A, and we call this set the operator spectrum of *A.*

Let the sequence $h : \mathbb{N} \to \mathbb{Z}^N$ tend to infinity, and let $A \in W(\mathbb{Z}^N)$ be as in (6). Then

$$(V_{-h(j)} A V_{h(j)} u)(x) = \sum_{\alpha\in\mathbb{Z}^N} a_\alpha(x + h(j))(V_\alpha u)(x).$$

Applying the Weierstrass-Bolzano theorem and a Cantor diagonal process, one gets a subsequence g of h such that, for every $\alpha \in \mathbb{Z}^N$, there is a function a_α^g with

$$a_\alpha(x + g(j)) \to a_\alpha^g(x), \quad x \in \mathbb{Z}^N.$$

Moreover,

$$\|a_\alpha^g\|_{l^\infty(\mathbb{Z}^N)} \le \|a_\alpha\|_{l^\infty(\mathbb{Z}^N)}. \tag{8}$$

Set

$$A_g := \sum_{\alpha\in\mathbb{Z}^N} a_\alpha^g V_\alpha.$$

It follows from (8) that A_g belongs to the Wiener algebra $W(\mathbb{Z}^N)$ again, and it has been proved in [20] that A_g is indeed the limit operator of A defined by the sequence g.

Proposition 4 *The operator spectrum $op\,(A|l^p(\mathbb{Z}^N))$ does not depend on $p \in (1, \infty)$ for $A \in W(\mathbb{Z}^N)$.*

Theorem 5 *Let $A \in W(\mathbb{Z}^N)$ and $p \in (1, \infty)$. Then $A : l^p(\mathbb{Z}^N) \to l^p(\mathbb{Z}^N)$ is a Fredholm operator if and only if there exists a $p_0 \in [1, \infty]$ such that all limit operators of A are invertible on $l^{p_0}(\mathbb{Z}^N)$.*

In particular this shows that the invertibility of all limit operators of A on $l^{p_0}(\mathbb{Z}^N)$ for some p_0 implies their invertibility on $l^p(\mathbb{Z}^N)$ for every p.

The following theorem is an important corollary of Theorem 5.

Theorem 6 *Let $A \in W(\mathbb{Z}^N)$. Then, for every $p \in (1, \infty)$,*

$$sp_{ess}\,(A|l^p(\mathbb{Z}^N)) = \bigcup_{A_g \in op\,(A|l^p(\mathbb{Z}^N))} sp\,(A_g|l^p(\mathbb{Z}^N)). \tag{9}$$

According to Propositions 2 and 4, neither the spectra $sp\,(A_g|l^p(\mathbb{Z}^N))$ nor the operator spectrum $op\,(A|l^p(\mathbb{Z}^N))$ do depend on $p \in (1, \infty)$. Hence, the following holds.

Theorem 7 *Let $A \in W(\mathbb{Z}^N)$. Then the essential spectrum of the operator A, considered as acting on $l^p(\mathbb{Z}^N)$, does not depend on $p \in (1, \infty)$.*

This theorem and the above remarks justify to write the equality (9) simply as

$$sp_{ess}\,(A) = \bigcup_{A_g \in op\,(A)} sp\,(A_g). \tag{10}$$

The following proposition gives a sufficient condition for the absence of the discrete spectrum of an operator A in the Wiener algebra.

Proposition 8 *Let $A \in W(\mathbb{Z}^N)$, and assume there exist a sequence $h \to \infty$ and an operator A_h such that*

$$\lim_{j\to\infty} \|V_{-h(j)} A V_{h(j)} - A_h\| = 0. \tag{11}$$

Then $sp_{ess}\,(A) = sp\,(A)$.

Proof. Let $\lambda \notin sp_{ess}\,(A)$. Then $A - \lambda I$ is a Fredholm operator. By Theorem 5, all limit operators $A_h - \lambda I$ are invertible. Hence, $\lambda \notin sp\,(A_h)$ for every limit operator A_h. It follows from (11) that $\lambda \notin sp\,(A)$. Hence, $sp\,(A) \subseteq sp_{ess}\,(A)$. ■

2.2 Slowly oscillating coefficients

A function $a \in l^\infty(\mathbb{Z}^N)$ is said to be *slowly oscillating at infinity* if

$$\lim_{x\to\infty} (a(x+y) - a(x)) = 0$$

for every $y \in \mathbb{Z}^N$. We denote the class of all slowly oscillating matrix-functions by $SO(\mathbb{Z}^N)$. If a is slowly oscillating then, for every sequence $h \to \infty$, there are a subsequence g of h and a complex number a^g such that

$$\lim_{j\to\infty} a(x + g(j)) = \lim_{j\to\infty} a(g(j)) = a^g$$

for each $x \in \mathbb{Z}^N$ (see [20]). Thus, every limit function of a slowly oscillating function is constant. We write $W^{SO}(\mathbb{Z}^N)$ for the subalgebra of $W(\mathbb{Z}^N)$ which

contains all operators with slowly oscillating coefficients. If $A \in W^{SO}(\mathbb{Z}^N)$, then all limit operators of A have the form

$$A^g := \sum_{\alpha \in \mathbb{Z}^N} a_\alpha^g V_\alpha, \quad \text{with } a_\alpha^g \in \mathbb{C}, \tag{12}$$

i.e., they are operators with constant diagonals.

For a complex-valued function u on $\mathbb{Z}^N$ with compact support, we define their discrete Fourier transform $Fu = \hat{u}$ by

$$\hat{u}(t) = (Fu)(t) := \frac{1}{(2\pi)^{N/2}} \sum_{x \in \mathbb{Z}^N} u(x) t^{-x}, \quad t \in \mathbb{T}^N,$$

where $\mathbb{T}$ is the unit circle on the complex plane, considered as a multiplicative group, $\mathbb{T}^N := \mathbb{T} \times \ldots \times \mathbb{T}$ is the N-dimensional torus, and $t^x := t_1^{x_1} \ldots t_N^{x_N}$ for $t \in \mathbb{T}^N$ and $x \in \mathbb{Z}^N$. Further, let

$$d\mu(t) := \frac{1}{(2\pi)^{N/2}\, i^N} \frac{dt_1 \ldots dt_N}{t_1 \ldots t_N}$$

be the normalized Haar measure on $\mathbb{T}^N$. Then the inverse Fourier transform is given by

$$u(x) = (F^{-1}\hat{u})(x) = \int_{\mathbb{T}^N} \hat{u}(t)\, t^x \, d\mu(t), \quad x \in \mathbb{Z}^N.$$

Moreover, the Fourier transform extends continuously to a unitary operator $F : l^2(\mathbb{Z}^N) \to L^2(\mathbb{T}^N, d\mu)$ which has F^{-1} as its adjoint.

The discrete Fourier transform establishes a unitary equivalence between the operator A^g in (12) and the operator of multiplication by the function

$$\hat{A}^g : \mathbb{T}^N \to \mathbb{C}, \quad t \mapsto \sum_{\alpha \in \mathbb{Z}^N} a_\alpha^g \, t^{-\alpha}.$$

In combination with (10), this yields the following.

Theorem 9 *Let $A \in W^{SO}(\mathbb{Z}^N)$. Then*

$$sp_{ess}\,(A) = \bigcup_{A_g \in op\,(A)} sp\,(A_g) = \bigcup_{A_g \in op\,(A)} \bigcup_{t \in \mathbb{T}^N} \{\hat{A}^g(t)\}.$$

2.3 Almost-periodic, r-periodic, and semi-periodic coefficients

A function $a \in l^\infty(\mathbb{Z}^N)$ is said to be *almost-periodic* if, for each sequence h tending to infinity, there are a subsequence g of h and a function $a^g \in l^\infty(\mathbb{Z}^N)$ such that

$$\lim_{j \to \infty} \|V_{-g(j)} a - a^g\|_{l^\infty(\mathbb{Z}^N)} = 0.$$

The almost-periodic functions form a C^*-subalgebra of $l^\infty(\mathbb{Z}^N)$ which we denote by $AP(\mathbb{Z}^N)$. We further write $W^{AP}(\mathbb{Z}^N)$ for the subalgebra of $W(\mathbb{Z}^N)$ which contains all operators with almost-periodic coefficients.

Proposition 10 *Let $A \in W^{AP}(\mathbb{Z}^N)$, and let A_h be the limit operator of A defined by a sequence $h \to \infty$. Then there is a subsequence g of h such that*

$$\lim_{j\to\infty} \|V_{-g(j)} A V_{g(j)} - A_h\| = 0. \tag{13}$$

Thus, A_h is a limit operator with respect to *norm convergence*. The proof follows immediately from the definition of the algebra $W^{AP}(\mathbb{Z}^N)$.

Corollary 11 *Let $A \in W^{AP}(\mathbb{Z}^N)$, and let A_h be a limit operator of A. Then*

$$sp\,(A) = sp_{ess}\,(A) \quad \textit{and} \quad sp\,(A) = sp\,(A_h). \tag{14}$$

Indeed, the first equality follows from Proposition 8, and the second one is a consequence of (13).

Let r be an N-tuple of positive integers. A function a on $\mathbb{Z}^N$ is called *r-periodic* if $a(x+r) = a(x)$ for each $x \in \mathbb{Z}^N$. The class of all r-periodic functions is denoted by $P_r(\mathbb{Z}^N)$. We further write $W^{P_r}(\mathbb{Z}^N)$ for the subalgebra of $W(\mathbb{Z}^N)$ which consists of all operators with r-periodic coefficients. Evidently,

$$W^{P_r}(\mathbb{Z}^N) \subset W^{AP}(\mathbb{Z}^N).$$

Hence, the equalities (14) hold for operators in $W^{P_r}(\mathbb{Z}^N)$ as well.

Let $r := (r_1, \ldots, r_N)$ and $d := r_1 \cdot \ldots \cdot r_N$, and write $\mathbb{C}^d$ as

$$\mathbb{C}^{r_1} \otimes \ldots \otimes \mathbb{C}^{r_N} =: \otimes_{j=1}^N \mathbb{C}^{r_j}.$$

Let $\{e_1, \ldots, e_N\}$ stand for the standard basis of $\mathbb{C}^N$, i.e., the jth entry of e_j is one, and the other entries are zero. Then the vectors $\{e_{j_1} \otimes \ldots \otimes e_{j_N}\}_{j_1=1, \ldots, j_N=1}^{r_1, \ldots, r_N}$ form a basis of $\mathbb{C}^d = \otimes_{j=1}^N \mathbb{C}^{r_j}$. We put these vectors into lexicographic order. Let

$$T_r : l^2(\mathbb{Z}^N) \to l^2(\mathbb{Z}^N, (\otimes_{j=1}^N \mathbb{C}^{r_j}))$$

be the unitary operator defined by

$$(T_r f)(y_1, \ldots, y_N) := (f(r_1 y_1 + j_1 - 1, \ldots, r_N y_N + j_N - 1))_{j_1=1, \ldots, j_N=1}^{r_1, \ldots, r_N}. \tag{15}$$

For $a \in l^\infty(\mathbb{Z}^N)$, set $\mu(a) := T_r a T_r^{-1}$. Clearly, $\mu(a)$ is given by the diagonal matrix

$$(\mathrm{diag}\,(a(j_1, \ldots, j_N)))_{j_1=1, \ldots, j_N=1}^{r_1, \ldots, r_N} \tag{16}$$

with the entries on the main diagonal being in lexicographic order. Further, one has

$$T_r(V_1^{\alpha_1} \ldots V_N^{\alpha_N}) T_r^{-1} = \Lambda_1^{-\alpha_1} \otimes \ldots \otimes \Lambda_N^{-\alpha_N}$$

where Λ_j is the $r_j \times r_j$-matrix operator

$$\Lambda_j = \begin{pmatrix} 0 & I & 0 & \cdot & 0 \\ 0 & 0 & I & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & I \\ V_j^{-1} & 0 & 0 & \cdot & 0 \end{pmatrix}. \tag{17}$$

Hence, the operator $A := \sum_{\alpha \in \mathbb{Z}^N} a_\alpha V_\alpha \in W^{P_r}(\mathbb{Z}^N)$ acting on $l^2(\mathbb{Z}^N)$ is unitarily equivalent to the operator

$$\mathcal{A} := T_r A T_r^{-1} = \sum_{\alpha \in \mathbb{Z}^N} \mu(a_\alpha)\, \Lambda_1^{-\alpha_1} \otimes \ldots \otimes \Lambda_N^{-\alpha_N}$$

acting on $l^2(\mathbb{Z}^N, \otimes_{j=1}^N \mathbb{C}^{r_j})$. Furthermore, the operator $\mathcal{A}$ is unitarily equivalent to the operator of multiplication by the continuous function $\sigma(\mathcal{A}) : \mathbb{T}^N \to \otimes_{j=1}^N \mathbb{C}^{r_j}$ which maps $t = (t_1, \ldots, t_N) \in \mathbb{T}^N$ to

$$\sigma(\mathcal{A})(t) := \sum_{\alpha=(\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}^N} \mu(a_\alpha)\, \hat{\Lambda}_1^{-\alpha_1}(t_1) \otimes \ldots \otimes \hat{\Lambda}_N^{-\alpha_N}(t_N) \tag{18}$$

where

$$\hat{\Lambda}_j(t_j) := \begin{pmatrix} 0 & 1 & 0 & \cdot & 0 \\ 0 & 0 & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 1 \\ t_j^{-1} & 0 & 0 & \cdot & 0 \end{pmatrix}.$$

Theorem 12 *Let $A \in W^{P_r}(\mathbb{Z}^N)$ and $p \in [1, \infty]$. The operator $A : l^p(\mathbb{Z}^N) \to l^p(\mathbb{Z}^N)$ is invertible if and only if*

$$\det \sigma(\mathcal{A})(t) \neq 0 \quad \textit{for each } t \in \mathbb{T}^N.$$

Corollary 13 *Let $A \in W^{P_r}(\mathbb{Z}^N)$ and $p \in [1, \infty]$. Then both the spectrum and the essential spectrum of $A : l^p(\mathbb{Z}^N) \to l^p(\mathbb{Z}^N)$ are independent of p, and*

$$sp_{ess}\,(A) = sp\,(A) = \cup_{t \in \mathbb{T}^N}\, sp\,(\sigma(\mathcal{A}(t))).$$

Let finally $SP_r(\mathbb{Z}^N)$ denote the smallest closed subalgebra of $l^\infty(\mathbb{Z}^N)$ which contains the algebras $SO(\mathbb{Z}^N)$ of the slowly oscillating functions and $P_r(\mathbb{Z}^N)$ of the r-periodic functions. Evidently, $SP_r(\mathbb{Z}^N)$ is a C^*-subalgebra of $l^\infty(\mathbb{Z}^N)$. We refer to the functions in this algebra as *semi-periodic functions*. Further, we write $W^{SP_r}(\mathbb{Z}^N)$ for the subalgebra of $W(\mathbb{Z}^N)$ of all operators with coefficients in $SP_r(\mathbb{Z}^N)$.

Proposition 14 *All limit operators of operators in $W^{SP_r}(\mathbb{Z}^N)$ belong to the algebra $W^{P_r}(\mathbb{Z}^N)$.*

Proof. A Cantor diagonal argument shows that it is sufficient to prove the assertion for diagonal operators in $W^{SP_r}(\mathbb{Z}^N)$, i.e., for operators of multiplication by functions in $SP_r(\mathbb{Z}^N)$. Another application of Cantor's diagonal argument implies furthermore that it is sufficient to prove the assertion for a dense subset of $SP_r(\mathbb{Z}^N)$.

Thus, let $a = \sum_{k=1}^m a_k b_k$ with $a_k \in P_r(\mathbb{Z}^N)$ and $b_k \in SO(\mathbb{Z}^N)$, and let h be a sequence tending to infinity. It is easy to see that there is a subsequence l of h such that

$$V_{-l(j)} a_k V_{l(j)} = a_k \quad \text{for all } j \text{ and } k.$$

Further, by what has already been said, there are a subsequence g of l and constants b_k^g such that

$$\text{s-lim}_{j\to\infty} V_{-g(j)} b_k V_{g(j)} = b_k^g I \quad \text{for each } k.$$

Thus, $\text{s-lim}_{j\to\infty} V_{-g(j)} a V_{g(j)}$ is equal to $a^g := \sum_{k=1}^m a_k b_k^g$ which is in $P_r(\mathbb{Z}^N)$. ■

Proposition 15 *Let $A \in W^{SP_r}(\mathbb{Z}^N)$ and $p \in [1, \infty]$. Then the essential spectrum of $A : l^p(\mathbb{Z}^N) \to l^p(\mathbb{Z}^N)$ is independent of p, and*

$$sp_{ess}(A) = \bigcup_{A_h \in op(A)} \bigcup_{t \in \mathbb{T}^N} sp\,(\sigma(\mathcal{A}_h(t))).$$

3 Schrödinger operators

The aim of this section is to describe the essential spectrum of discrete Schrödinger operators of the form

$$H := \sum_{k=1}^N (V_{e_k} - a_k I)\,(V_{-e_k} - \bar{a}_k I) + \Phi I \tag{19}$$

where $e_j := (0, \ldots, 0, 1, \ldots, 0) \in \mathbb{Z}^N$ with the 1 standing at the jth position, the a_j are bounded complex-valued functions on $\mathbb{Z}^N$, and Φ is a bounded real-valued function on $\mathbb{Z}^N$. The vector $a := (a_1, \ldots, a_N)$ can be viewed of as the discrete analog of the magnetic potential, whereas Φ serves as the discrete analog of the electric potential. Clearly, H acts as a self-adjoint operator on the Hilbert space $l^2(\mathbb{Z}^N)$.

3.1 Slowly oscillating potentials

We suppose that the vector potential $a = (a_1, \ldots, a_N)$ and the scalar potential Φ are slowly oscillating at infinity. Moreover, we assume that

$$\lim_{x\to\infty} |a_j(x)| = 1 \quad \text{for each } j = 1, \ldots, N. \tag{20}$$

Under these assumptions, all limit operators H_g of H are of the form

$$H_g = \sum_{k=1}^N (V_{e_k} - a_k^g I)\,(V_{-e_k} - \bar{a}_k^g) + \Phi^g I$$

where the a_k^g are complex constants and the Φ^g are real constants given by the pointwise limits

$$a_k^g = \lim_{m\to\infty} a_k(x + g(m)) \quad \text{and} \quad \Phi^g = \lim_{m\to\infty} \Phi(x + g(m)).$$

Moreover, $|a_k^g| = 1$. Hence, the limit operators H_g can be written as Schrödinger operators of the form

$$H_g = \sum_{k=1}^{N} (2 - a_k^g V_{-e_k} - \bar{a}_k^g V_{e_k}) + \Phi^g I.$$

Write a_k^g as $e^{i\varphi_k^g}$ with $\varphi_k^g \in [0, 2\pi)$ and set $\varphi^g = (\varphi_1^g, \ldots, \varphi_N^g)$. Consider the unitary operator $U : l^2(\mathbb{Z}^N) \to l^2(\mathbb{Z}^N)$ defined by

$$(Uv)(x) := e^{-i\langle \varphi^g, x\rangle} v(x).$$

Then

$$\tilde{H}_g := U H_g U^* = \sum_{k=1}^{N} (2I - V_{-e_k} - V_{e_k}) + \Phi^g I.$$

The operator $\tilde{H}_g$ is unitarily equivalent to the operator of multiplication by the function

$$[0, 2\pi)^N \to \mathbb{C}, \quad (\psi_1, \ldots, \psi_N) \mapsto 4 \sum_{k=1}^{N} \sin^2 \frac{\psi_k}{2} + \Phi^g,$$

thought of as acting on $L^2(\mathbb{T}^N)$. Hence,

$$sp\,(\tilde{H}_g) = \left\{ 4 \sum_{k=1}^{N} \sin^2 \frac{\psi_k}{2} + \Phi^g : \psi_k \in [0, 2\pi) \right\} = [\Phi^g, \Phi^g + 4N]. \tag{21}$$

Since the set of all partial limits of a slowly oscillating function is connected (see Thorem 2.4.7 in [22] or [19], for instance [19]), formula (10) implies

$$sp_{ess}\,(H) = \bigcup [\Phi^g, \Phi^g + 4N] = [m(\Phi), M(\Phi) + 4N] \tag{22}$$

where the union is taken over all sequences g for which the limit operator H_g exists, and where

$$m(\Phi) := \liminf_{x \to \infty} \Phi(x) \quad \text{and} \quad M(\Phi) := \limsup_{x \to \infty} \Phi(x).$$

Hence, *the essential spectrum of the Schrödinger operator* (*19*) *does not depend on slowly oscillating magnetic potentials.*

3.2 Periodic and semi-periodic potentials

Next we consider the Schrödinger operator H in (19) with r-periodic coefficients, that is, the a_k as well as the potential Φ are r-periodic functions. The unitary operator $T_r : l^2(\mathbb{Z}^N) \to l^2(\mathbb{Z}^N, \mathbb{C}^d)$ defined by (15) induces a unitary equivalence between the operator H and the matrix operator

$$\mathcal{H} := \sum_{k=1}^{N} (\Lambda_{e_k} - \mu(a_k))\,(\Lambda_{-e_k} - \overline{\mu(a_k)}) + \mu(\Phi),$$

acting on the space $l^2(\mathbb{Z}^N, \mathbb{C}^d)$. Again, the Λ_{e_k} are defined by (17), and the $\mu(a_k)$ and $\mu(\Phi)$ are given by (16). The operator $\mathcal{H}$ on its hand is unitarily equivalent to the operator of multiplication by the Hermitian matrix function

$$\sigma_{\mathcal{H}}(t) = \sum_{k=1}^{N} (\hat{\Lambda}_{e_k}(t_k) - \mu(a_k))\,(\hat{\Lambda}_{-e_k}(t_k) - \overline{\mu(a_k)}) + \mu(\Phi), \quad t \in \mathbb{T}^N,$$

acting on the space $L^2(\mathbb{T}^N, \mathbb{C}^d)$. Let $\lambda_1^H(t) \leq \ldots \leq \lambda_d^H(t)$ denote the eigenvalues of the matrix $\sigma_{\mathcal{H}}(t)$. The functions $t \mapsto \lambda_k^{\mathcal{H}}(t)$ are continuous on $\mathbb{T}^N$ (see [10]). Since continuous functions map compact connected sets into compact connected sets again, the sets

$$\Gamma_k^H := \{\lambda_k^{\mathcal{H}}(t) : t \in \mathbb{T}^N\} \tag{23}$$

are closed subintervals of $\mathbb{R}$. Hence, *the spectrum of the periodic Schrödinger operator (19) coincides with its essential spectrum, and it can be represented as the union of finitely many real intervals.*

Let now H be a Schrödinger operator of the form (19) with semi-periodic coefficients. Thus, the coefficients a_k and the potential Φ belong to $SP_r(\mathbb{Z}^N)$. Then all limit operators H^g of H are operators with r-periodic coefficients. Hence, the essential spectrum of H is equal to

$$sp_{ess}\,(H) := \bigcup_{H^g \in op\,(H)} \bigcup_{k=1}^{d} \Gamma_k^{H^g}$$

with the sets $\Gamma_k^{H^g}$ defined by (23).

We conclude this section by an example which shows that a Schrödinger operator with periodic coefficients can have gaps in its spectrum, whereas a slowly oscillating perturbation of that operator can close the gaps. More precisely, we consider the one-dimensional Schrödinger operator

$$H := V_{-1} + V_1 + (\Phi + \Psi)I$$

where Φ is a 2-periodic real-valued function on $\mathbb{Z}$ and Ψ belongs to $SO(\mathbb{Z})$. All limit operators of H are of the form

$$H^g = V_{-1} + V_1 + (\Phi^g + \Psi^g)I,$$

where

$$\Phi^g(x) = \lim_{k\to\infty} \Phi(x + g(k))$$

is a 2-periodic real-valued function and

$$\Psi^g = \lim_{k\to\infty} \Psi(x + g(k))$$

is a real constant. It follows from (14) that

$$sp\,(V_{-1} + V_1 + \Phi^g I) = sp\,(V_{-1} + V_1 + \Phi I).$$

Hence,

$$sp\,(H^g) = sp\,(V_{-1} + V_1 + \Phi I) + \Psi^g.$$

Let $\Psi = 0$ for a moment. The operator $V_{-1} + V_1 + \Phi I$ is unitarily equivalent to the operator of multiplication by the matrix function

$$\mathcal{A}(t) := \begin{pmatrix} \Phi(0) & 1+t \\ 1+t^{-1} & \Phi(1) \end{pmatrix}, \quad t \in \mathbb{T}.$$

The eigenvalues of $\mathcal{A}(t)$ are the solutions of the equation

$$\lambda^2 - (\Phi(0) + \Phi(1))\lambda + \Phi(0)\Phi(1) - \gamma(t) = 0 \tag{24}$$

where $\gamma(t) := 2 + t + t^{-1}$. Let $|\Phi(0) - \Phi(1)| > 2$. Then (24) has two different solutions $\lambda_1(t) < \lambda_2(t)$ for every $t \in \mathbb{T}$, and λ_1, λ_2 are smooth real-valued functions on $\mathbb{T}$. Set $m_j := \min_{t\in\mathbb{T}} \lambda_j(t)$ and $M_j := \max_{t\in\mathbb{T}} \lambda_j(t)$ for $j = 1, 2$. Then

$$sp\,(V_{-1} + V_1 + \Phi I) = [m_1,\, M_1] \cup [m_2,\, M_2].$$

Thus, in case $M_1 < m_2$, the operator $V_{-1} + V_1 + \Phi I$ has a gap $(M_1,\, m_2)$ in its spectrum.

Now consider the slowly oscillating perturbation $H := V_{-1} + V_1 + (\Phi + \Psi)I$ of the operator $V_{-1}+V_1+\Phi I$. The essential spectrum of the perturbed operator H is union of the spectra of its limit operators H^g. It is evident from what has been said above that

$$sp\,(H^g) = [m_1 + \Psi^g,\, M_1 + \Psi^g] \cup [m_2 + \Psi^g,\, M_2 + \Psi^g].$$

Since the set of all partial limits of a slowly oscillating function is connected, we conclude that

$$\begin{aligned} sp_{ess}\,(H) &= \bigcup_{H^g \in op\,(H)} [m_1 + \Psi^g,\, M_1 + \Psi^g] \cup [m_2 + \Psi^g,\, M_2 + \Psi^g] \\ &= [m_1 + m(\Psi),\, M_1 + M(\Psi)] \cup [m_2 + m(\Psi),\, M_2 + M(\Psi)], \end{aligned}$$

where $m(\Psi) := \liminf_{x\to\infty} \Psi(x)$ and $M(\Psi) := \limsup_{x\to\infty} \Psi(x)$. Thus, if

$$M(\Psi) - m(\Psi) \geq m_2 - M_1,$$

then the perturbed operator H has no gaps in its essential spectrum, and

$$sp_{ess}\,(H) = [m_1 + m(\Psi),\, M_2 + M(\Psi)]$$

in this case.

3.3 Discrete quantum waveguides

Here we are going to examine the essential spectrum of the Schrödinger operator

$$H = \sum_{j=1}^{N} (2I - V_{e_j} - V_{-e_j}) + \Phi I.$$

under the assumption that the electric potential Φ has the form

$$\Phi(x) := \chi_+(x_N)\Phi_+(x) + \chi_0(x_N)\Phi_0(x) + \chi_-(x_N)\Phi_-(x), \quad x \in \mathbb{Z}^N$$

where χ_+, χ_- and χ_0 are equal to 1 on the intervals (h_2, ∞), $(-\infty, h_1)$ and $[h_1, h_2]$ and equal to 0 outside these intervals, respectively, and where $\Phi_\pm$ and Φ_0 are function in $SO(\mathbb{Z}^N)$. Here, $h_1 < h_2$ are previously fixed real numbers.

To describe the essential spectrum of H we consider the limit operators of H defined by sequences $g = (g', g_N) : \mathbb{N} \to \mathbb{Z}^{N-1} \times \mathbb{Z}$ tending to infinity. We have to distinguish between two cases.

In the first case, we assume that the sequence $g = (g', g_N)$ is such that $g_N \to \pm\infty$. In this case, the limit operators of H are of the form

$$H_g^\pm = \sum_{j=1}^{N}(2 - V_{e_j} - V_{-e_j}) + \Phi_\pm^g I$$

where the $\Phi_\pm^g := \lim_{j\to\infty} \Phi_\pm(g(j))$ are real numbers. Consequently,

$$sp\,(H_g^\pm) = [\Phi_\pm^g,\, \Phi_\pm^g + 4N],$$

as mentioned in the previous section.

In the second case, we assume that $g' \to \infty$ whereas g_N is a constant sequence. Then the limit operators of H are

$$H_g = \sum_{j=1}^{N}(2 - V_{e_j} - V_{-e_j}) + \Phi^g I \tag{25}$$

where the function Φ^g depends on x_N only. Moreover, this function is piecewise constant since

$$\Phi^g(x_N) = \chi_+(x_N)\Phi_+^g + \chi_0(x_N)\Phi_0^g + \chi_-(x_N)\Phi_-^g$$

with real numbers $\Phi_\pm^g := \lim_{j\to\infty} \Phi_\pm(g(j))$ and $\Phi_0^g := \lim_{j\to\infty} \Phi_0(g(j))$.

The operator (25) is unitarily equivalent to the operator of multiplication by the operator-valued function $\hat{H}_g : \mathbb{T}^{N-1} \to L(l^2(\mathbb{Z}))$ defined by

$$\hat{H}_g(t') := \sum_{j=1}^{N-1}(2 - t_j - t_j^{-1})I + (2 - V_{e_N} - V_{-e_N}) + \Phi^g I$$

where $t' = (t_1, \ldots, t_{N-1})$. It is well-known (see [17, 25], for instance) that spectrum of the one-dimensional Jacobi operator

$$\mathcal{L}_N^g := (2 - V_{e_N} - V_{-e_N}) + \Phi^g I$$

is the union of its essential spectrum $\Sigma_g := [\Phi_+^g,\, 4 + \Phi_+^g] \cup [\Phi_-^g,\, 4 + \Phi_-^g]$ with a finite set $\{\lambda_1^g, \ldots, \lambda_{m(g)}^g\}$ of points in the discrete spectrum which are located outside Σ_g. Hence,

$$sp\,(\mathcal{L}_N^g) = [\Phi_+^g,\, 4N + \Phi_+^g] \bigcup [\Phi_-^g,\, 4N + \Phi_-^g] \bigcup_{j=1}^{m(g)} [\lambda_j^g,\, 4(N-1)].$$

Set $\mathbb{Z}_+^N := \{z \in \mathbb{Z}^N : z_N > 0\}$ and $\mathbb{Z}_-^N := \mathbb{Z}^N \setminus \mathbb{Z}_+^N$, and abbreviate

$$m(\Phi_\pm) := \liminf_{\mathbb{Z}_\pm^N \ni x \to \infty} \Phi_\pm, \quad M(\Phi_\pm) := \limsup_{\mathbb{Z}_\pm^N \ni x \to \infty} \Phi_\pm.$$

Then equality (10) implies that $sp_{ess}\,(H)$ is equal to

$$[m(\Phi_-),\, M(\Phi_-) + 4N] \bigcup [m(\Phi_+),\, M(\Phi_+) + 4N] \bigcup_{g \in \Lambda} \bigcup_{j=1}^{m(g)} [\lambda_j^g,\, 4(N-1)]$$

where Λ refers to the set of all sequences $g = (g',\, g_N)$ for which $g' \to \infty$ and g_N is constant.

3.4 Potentials with an infinite set of discontinuities

Set $\Delta := 2I - V_{-1} - V_1 \in L(l^2(\mathbb{Z}))$. We consider the Schrödinger operator

$$H = \Delta + \Phi I : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$$

where the potential Φ takes two real values a, b only, but where Φ is allowed to have an infinite set of jumps. More precisely, let $\Phi(x) = a$ if $x \in \Lambda$ and let $\Phi(x) = b$ if $x \in \mathbb{Z} \setminus \Lambda$ where

$$\Lambda := \cup_{k=0}^{\infty} \{x \in \mathbb{Z} : \gamma_k^- \le |x| \le \gamma_k^+\}$$

and where the γ_k^- and γ_k^+ are integers satisfying

$$\lim_{k\to\infty} \gamma_k^- = \lim_{k\to\infty} \gamma_k^+ = \lim_{k\to\infty} (\gamma_k^+ - \gamma_k^-) = \lim_{k\to\infty} (\gamma_{k+1}^- - \gamma_k^+) = +\infty. \tag{26}$$

For example, this condition holds if $\gamma_k^- := k^2$ and $\gamma_k^+ := k^2 + k$.

Theorem 16 *Under these assumptions,* $sp_{ess}\,(\Delta + \Phi I) = [a,\, a+4] \cup [b,\, b+4]$.

Proof. First we determine all limit operators of the operator ΦI with respect to sequences g tending to $+\infty$. Again, we have to distinguish between several possibilities.

Case a) Let the sequence $g \to +\infty$ satisfy

$$\lim_{k\to\infty} (\gamma_k^+ - g(k)) = \lim_{k\to\infty} (g(k) - \gamma_k^-) = +\infty.$$

Such sequences exist because of $\gamma_k^+ - \gamma_k^- \to +\infty$.

In this case, for every $x \in \mathbb{Z}$, there exists a $k(x)$ with $x + g(k) \in (\gamma_k^-,\, \gamma_k^+)$ for each $k \ge k(x)$. Thus, $\lim_{k\to\infty} \Phi(x + g(k)) = a$ for every $x \in \mathbb{Z}$, whence

$$\text{s-lim}_{k\to\infty} V_{-g(k)} \Phi V_{g(k)} = aI.$$

Case b) Let the sequence $g \to +\infty$ be such that

$$\lim_{k\to\infty} (\gamma_{k+1}^- - g(k)) = \lim_{k\to\infty} (g(k) - \gamma_k^+) = +\infty.$$

Such sequences exist because of $\gamma_{k+1}^- - \gamma_k^+ \to +\infty$.

Under this assumption, for every $x \in \mathbb{Z}$, one finds a $k(x)$ such that $x+g(k) \in (\gamma_k^+, \gamma_{k+1}^-)$ for each $k \geq k(x)$. Thus, $\lim_{k\to\infty} \Phi(x+g(k)) = b$ for every $x \in \mathbb{Z}$, whence

$$\text{s-lim}_{k\to\infty} V_{-g(k)} \Phi V_{g(k)} = bI.$$

Case c) Let now $g(k) = \gamma_k^- - h(k)$ with a bounded sequence h. Passing to a suitable subsequence, one can assume that $g(k) = \gamma_k^- - h$ where h is constant. If $x - h < 0$, then $\lim_{k\to\infty} \Phi(x+g(k)) = b$, whereas this limit is equal to a in case $x - h \geq 0$. We denote the characteristic function of the set $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$ by χ_+ and set $\chi_- := 1 - \chi_+$. Then

$$\lim_{k\to\infty} \Phi(x+g(k)) = a\chi_+(x-h) + b\chi_-(x-h), \quad \text{for } x \in \mathbb{Z},$$

which implies that

$$\text{s-lim}_{k\to\infty} V_{-g(k)} \Phi V_{g(k)} = (a\chi_+^h + b\chi_-^h)I$$

with $\chi_\pm^h(x) := \chi_\pm(x-h)$.

Case d) Let finally $g(k) = \gamma_k^+ - h(k)$ where h is a bounded sequence. Passing to a suitable subsequence one can assume again that $g(k) = \gamma_k^- - h$ where h is constant. Now one has $\lim_{k\to\infty} \Phi(x+g(k)) = a$ for $x - h < 0$, and this limit is equal to b for $x - h \geq 0$. Consequently,

$$\lim_{k\to\infty} \Phi(x+g(k)) = a\chi_-^h(x) + b\chi_+^h(x), \quad \text{for } x \in \mathbb{Z}$$

with $\chi_\pm^h$ as above. This shows that

$$\text{s-lim}_{k\to\infty} V_{-g(k)} \Phi V_{g(k)} = (a\chi_-^h + b\chi_+^h)I.$$

Evidently, every sequence g tending to $+\infty$ is subject to one of these four cases. Thus, the set of the limit operators of ΦI defined by sequences tending to $+\infty$ is exhausted by the operators aI, bI, $(a\chi_+^h + b\chi_-^h)I$ and $(a\chi_-^h + b\chi_+^h)I$ where h runs through $\mathbb{Z}$. Similarly one checks that the set of all limit operators of ΦI defined by sequences g tending to $-\infty$ consists exactly of the same operators aI, bI, $(a\chi_+^h + b\chi_-^h)I$ and $(a\chi_-^h + b\chi_+^h)I$. Notice further that the operators $(a\chi_+^h + b\chi_-^h)I$ and $(a\chi_-^h + b\chi_+^h)I$ are unitarily equivalent to the operators $(a\chi_+^0 + b\chi_-^0)I = (a\chi_+ + b\chi_-)I$ and $(a\chi_-^0 + b\chi_+^0)I = (a\chi_- + b\chi_+)I$, respectively.

It is evident that the continuous spectra of the operators $\Delta + aI$ and $\Delta + bI$ with constants a, b are the intervals $[a, a+4]$ and $[b, b+4]$, respectively, whereas both the spectrum of the operator $\Delta + (a\chi_+ + b\chi_-)I$ and that of $\Delta + (a\chi_- + b\chi_+)I$ are equal to $[a, a+4] \cup [b, b+4]$ (see [17], for instance). Hence, the assertion follows from Theorem 6. ∎

The method of limit operators can be also applied to more involved potentials Φ of the form

$$\Phi = a\chi_\Lambda + b\chi_{\mathbb{Z}\setminus\Lambda} \tag{27}$$

where χ_Λ is the characteristic function of a subset Λ of $\mathbb{Z}$ as before, and where $a, b \in SO(\mathbb{Z})$.

Theorem 17 *Let Φ be a potential of the form (27). Then*

$$sp_{ess}\,(\Delta + \Phi I) = [m(a),\, M(a) + 4] \cup [m(b),\, M(b) + 4] \tag{28}$$

where $m(f) := \liminf_{x\to\infty} f(x)$ and $M(f) := \limsup_{x\to\infty} f(x)$ for each slowly oscillating function f.

Proof. We proceed as in the proof of the previous theorem to find that all limit operators of ΦI are unitarily equivalent to one of the operators

$$a^g I,\ b^g I,\ (a^g\chi_- + b^g\chi_+)I,\ (a^g\chi_+ + b^g\chi_-)I$$

with constants

$$a^g := \lim_{k\to\infty} a(g(k)) \quad \text{and} \quad b^g := \lim_{k\to\infty} b(g(k)). \tag{29}$$

Thus, according to Theorems 16 and 6,

$$sp_{ess}\,(H) = \bigcup([a^g,\, a^g + 4] \cup [b^g,\, b^g + 4])$$

where the union is taken over all sequences $g \to \infty$ for which the partial limits (29) of the functions a and b exist. Employing the connectedness of the set of all partial limits of a slowly oscillating function, we arrive at identity (28). ■

3.5 The three-particle problem

We consider the three-particle Schrödinger operator on $l^2(\mathbb{Z}^3 \times \mathbb{Z}^3)$

$$\begin{aligned} H \quad := \quad & \frac{1}{2m_1}(\Delta \otimes I) + \frac{1}{2m_2}(I \otimes \Delta) \\ & + (W_1 I) \otimes I + I \otimes (W_2 I) + W_{12}^{dif}(I \otimes I) \end{aligned} \tag{30}$$

which describes the motion on the lattice $\mathbb{Z}^3$ of two particles with coordinates $x^1,\, x^2 \in \mathbb{Z}^3$ with masses $m_1,\, m_2$ around a heavy nuclei located at the point 0. In (30) we write I for the identity operator on $l^2(\mathbb{Z}^3)$ and Δ for the discrete Laplacian

$$\Delta := \sum_{j=1}^{3}(2I - V_{-e_j} - V_{e_j})$$

on $l^2(\mathbb{Z}^3)$ where $\{e_1,\, e_2,\, e_3\}$ stands for the standard basis of $\mathbb{Z}^3$. Further, $W_1,\, W_2,\, W_{12}$ are real-valued functions on $\mathbb{Z}^3$ with

$$\lim_{z\to\infty} W_1(z) = \lim_{z\to\infty} W_2(z) = \lim_{z\to\infty} W_{12}(z) = 0,$$

and W_{12}^{dif} is the function on $\mathbb{Z}^3 \times \mathbb{Z}^3$ defined by

$$W_{12}^{dif}(x_1,\, x_2) := W_{12}(x_1 - x_2).$$

Further we abbreviate $m := 6/m_1 + 6/m_2$.

In order to describe the essential spectrum of the operator H by means of (10), we have to determine the limit operators of H. Let $g = (g^1, g^2) : \mathbb{N} \to \mathbb{Z}^3 \times \mathbb{Z}^3$ be a sequence tending to infinity. The examination of the limit operators leads to the consideration of the following situations.

Case 1) The sequence g^1 tends to infinity, whereas g^2 is constant. Then the limit operator H_g of H is unitarily equivalent to the operator

$$H_2 := \frac{1}{2m_1}(\Delta \otimes I) + \frac{1}{2m_2}(I \otimes \Delta) + I \otimes (W_2 I). \tag{31}$$

Case 2) If $g^2 \to \infty$ and g^1 is constant, then the limit operator H_g of H is unitarily equivalent to the operator

$$H_1 := \frac{1}{2m_1}(\Delta \otimes I) + \frac{1}{2m_2}(I \otimes \Delta) + (W_1 I) \otimes I. \tag{32}$$

Case 3a) Let both $g^1 \to \infty$ and $g^2 \to \infty$, and assume that also $g^1 - g^2 \to \infty$. In this case, the associated limit operator of H is the free discrete Hamiltonian $\frac{1}{2m_1}(\Delta \otimes I) + \frac{1}{2m_2}(I \otimes \Delta)$. The spectrum of this operator is the intervall $[0, m]$.

Case 3b) Let again $g^1 \to \infty$ and $g^2 \to \infty$, but now let $g^1 - g^2$ be a bounded sequence. Then $g^1 - g^2$ hac a constant subsequence; so we can assume without loss that the sequence $g^1 - g^2$ itself is constant. In this case, the limit operator with respect to g is unitarily equivalent to the operator of interaction of the particles x^1, x^2 which is given by

$$H_{12} := \frac{1}{2m_1}(\Delta \otimes I) + \frac{1}{2m_2}(I \otimes \Delta) + W_{12}^{dif}(I \otimes I). \tag{33}$$

It follows from Proposition 8 that the discrete spectra of the operators H_1, H_2 and H_{12} are empty. Thus, since the spectrum of each operator H_1, H_2 and H_{12} contains the interval $[0, m]$ with m as above, and due to (10),

$$sp_{ess}\,(H) = sp\,(H_1) \cup sp\,(H_2) \cup sp\,(H_{12}). \tag{34}$$

To determine the spectrum of H_2, we apply the Fourier transform with respect to the first variable. Then H_2 becomes unitarily equivalent to the operator of multiplication by the operator-valued function

$$\mathbb{T}^3 \to L(l^2(\mathbb{Z}^3)), \quad t \mapsto \sum_{j=1}^{3} \frac{1}{2m_1}(2 - t_j - t_j^{-1})I + \frac{1}{2m_2}\Delta + W_2 I. \tag{35}$$

The operator $\frac{1}{2m_2}\Delta + W_2 I$ acting on $l^2(\mathbb{Z}^3)$ has the essential spectrum

$$\frac{1}{2m_2}[0, 12] = [0, 6/m_2]$$

and a real discrete spectrum $\{\lambda_k^{(2)}\}_{k=1}^{\infty}$ which is located outside the interval $[0,\, 6/m_2]$ and which has 0 and $6/m_2$ as only possible accumulation points. Consequently,

$$sp\,(H_2) = [0,\, m] \bigcup_{k=1}^{\infty} [\lambda_k^{(2)},\, \lambda_k^{(2)} + 6/m_2]. \tag{36}$$

In the same way one obtains

$$sp\,(H_1) = [0,\, m] \bigcup_{k=1}^{\infty} [\lambda_k^{(1)},\, \lambda_k^{(1)} + 6/m_1] \tag{37}$$

where $\{\lambda_k^{(1)}\}_{k=1}^{\infty}$ is the sequence of the points of the discrete spectrum of $\frac{1}{2m_1}\Delta + W_1 I$ on $l^2(\mathbb{Z}^3)$ which are located outside $[0,\, 6/m_1]$ and which can accumulate only at 0 and 12.

Note that, unlike the continuous case of operators on $\mathbb{R}^6$, there is no transformation of the discrete operator H_{12} to an operator of the form $\Delta \otimes I + I \otimes \Delta + (W_{12} I) \otimes I$ with a multiplication operator $W_{12} I$ on $l^2(\mathbb{Z}^3)$. This fact makes the spectral theory of discrete operators of interaction much more difficult than the corresponding theory for continuous operators of interaction on $\mathbb{R}^3$ (see, for instance, [1, 2, 11]).

We can only give a simple estimate for the location of the spectrum of H_{12} which results from the following well-known estimate for the spectrum of a self-adjoint operator acting on a Hilbert space ([13], p. 357).

Proposition 18 *Let A be a self-adjoint operator on a Hilbert space. Then $sp\,(A) \subseteq [a,\, b]$ where*

$$a := \inf_{\|h\|=1} \langle Ah,\, h\rangle, \quad b := \sup_{\|h\|=1} \langle Ah,\, h\rangle.$$

The points a, b belong to the spectrum of A.

Proposition 18 implies the following estimates for the spectra of H_1, H_2 and H_{12}:

$$[0,\, m] \subseteq sp\,(H_j) \subseteq \left[\inf_{x\in\mathbb{Z}^3} W_j(x),\, \sup_{x\in\mathbb{Z}^3} W_j(x) + m\right]$$

for $j = 1,\, 2$, and

$$[0,\, m] \subseteq sp\,(H_{12}) \subseteq \left[\inf_{x\in\mathbb{Z}^3} W_{12}(x),\, \sup_{x\in\mathbb{Z}^3} W_{12}(x) + m\right].$$

In combination with (34), these inclusions yield lower and upper bounds for the essential spectrum of H.

Theorem 19 *Let H be the Schrödinger operator (30) on $l^2(\mathbb{Z}^3 \times \mathbb{Z}^3)$. Then*

$$\inf sp_{ess}\,(H) = \min\left(\inf_{x\in\mathbb{Z}^3} W_1(x),\, \inf_{x\in\mathbb{Z}^3} W_2(x),\, \inf_{x\in\mathbb{Z}^3} W_{12}(x)\right),$$

$$\sup sp_{ess} H = \max \left(\sup_{x \in \mathbb{Z}^3} W_1(x) + m,\ \sup_{x \in \mathbb{Z}^3} W_2(x) + m,\ \sup_{x \in \mathbb{Z}^3} W_{12}(x) + m \right)$$

with $m := 6/m_1 + 6/m_2$.

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